Solutions Sample   Oct 15, 14

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(a) $\int \frac{x\,dx}{4+9x^4}$ $u = 3x^2$ $du = 6x\,dx$.

$$= \int \frac{du}{6\left[4+u^2\right]} = \frac{1}{6} * \frac{1}{2} \tan^{-1}\frac{u}{2} + C.$$

$$= \frac{1}{12} \tan^{-1}\left(\frac{3x^2}{2}\right) + C.$$

(b) $\int \cos^{-1} x\,dx$ Method 1 = By parts.

Method 2 $\cos^{-1} x = \alpha$

$\Rightarrow \cos\alpha = x$.

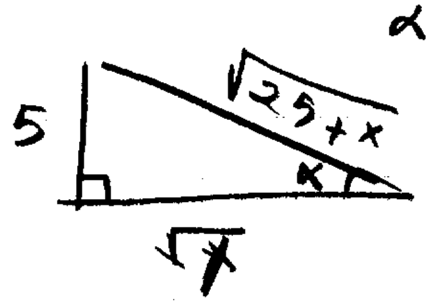
$-\sin\alpha\, d\alpha = dx$.

$\Rightarrow$ int $= -\int \alpha \cdot \sin\alpha\, d\alpha$ By parts

...

2 $\cos(\underbrace{\tan^{-1}(5/\sqrt{x})}_{\alpha})$; $\tan^{-1}\frac{5}{\sqrt{x}} = \alpha$

$\tan\alpha = \frac{5}{\sqrt{x}} = \frac{opp}{adj}$.



$\Rightarrow$ Need $\cos\alpha = \frac{adj}{hyp}$

$$= \frac{\sqrt{x}}{\sqrt{25+x}}$$

(b) $10\cosh(\ln x) = 10\left[\frac{e^{\ln x} - e^{-\ln x}}{2}\right]$

$= 5\left[x - \frac{1}{x}\right]$

(a) $\int_0^{\infty} \frac{2\,dx}{4+x^2} = \lim_{t\to\infty} \int_0^{t} \frac{2\,dt}{4+x^2}$

$\lim_{t\to\infty} 2 * \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)\Big|_0^t = \tan^{-1}\infty - \tan^{-1} 0$

$= \pi/2 - 0 = \pi/2$

(b) $\int_8^{\boxed{\infty}} \frac{dx}{(x-7)(x-6)} = \int\left(\frac{A}{x-7} + \frac{B}{(x-6)}\right) dx$

$= \lim_{t\to\infty} \int_8^t \left(\frac{1}{x-7} - \frac{1}{x-6}\right) dx = \lim_{t\to\infty}\left(\ln \frac{x-7}{x-6}\Big|_8^t\right)$

$\longrightarrow \ln 1 - \ln 1/2 \longrightarrow \boxed{\ln 2}$

(3) a : $\int_1^{\infty} \frac{\sqrt{x+2}}{x^5}\,dx \overset{LCT}{\approx} \int_1^{\infty} \frac{\sqrt{x}}{x^5}\,dx = \int_1^{\infty} \frac{1}{x^{4.5}}\,dx$

p-int ; $p > 1 \Rightarrow$ conv.

(b) $\int_0^{\boxed{3}} \frac{dx}{\sqrt{3-x}} = \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{3-x}}$ $\quad u = 3-x$ $\quad du = -dx$

$\int u^{-1/2}\,du = 2\sqrt{u}$

$$\lim_{t \to 3^-} -2\sqrt{3-x}\Big|_0^t = 2\sqrt{3}$$

$\Rightarrow$ converge.

(c) $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$ $\qquad \dfrac{\sqrt{x}}{1+x^2} \approx \dfrac{1}{x^{1.5}}$

$$= \boxed{\int_0^1 \cdots} + \boxed{\int_1^\infty \frac{\sqrt{x}}{1+x^2}} = K + \text{conv.}$$

$\Rightarrow$ conv.

proper $\Rightarrow \boxed{K}$ $\qquad \approx \dfrac{1}{x^{1.5}}$ conv.

(d) $\displaystyle\int_1^\infty \frac{|\ln x|}{x}\,dx > \int_1^\infty \frac{1}{x}\,dx$ div.

since $|\ln x| > 1$

$\Rightarrow$ Our integral $>$ div. $\Rightarrow$ Div. by DCT.

(e) $\displaystyle\int_1^\infty \left(\frac{\ln x}{x+\sqrt{x-1}}\right)dx \approx \int_1^\infty \frac{\ln x}{x}\,dx$ $\boxed{\text{Refer to (d)}}$

(f) $\displaystyle\int_1^\infty \frac{e^x}{\sqrt{1+x^2}}\,dx = \int_1^\infty \infty\,dx \Rightarrow$ div.

$\to \infty$

$\dfrac{e^x}{\sqrt{1+x^2}} = \dfrac{\text{fast}}{\text{slow}} \to \infty$

(g) $\displaystyle\int_0^\infty \frac{|\sin x|}{1+x^2}\,dx < \int_0^\infty \frac{1}{1+x^2}\,dx$ = conv. by DCT

$\Rightarrow$

(4) $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p} = \int_{\dots}^\infty \frac{du}{u^p}$

$u = \ln x \qquad du = \dfrac{dx}{x} \implies$ p-int. $\implies$

conv. for $p > 1$.

(5) (a) $\displaystyle\int_{-\infty}^{-3} \frac{8}{x^3}\,dx$ Since odd functn

$\displaystyle = \int_3^\infty -\frac{8}{x^3}\,dx \overset{LCT}{\approx} \int_3^\infty \frac{dx}{x^3}$ p-int. $p>1$ $\implies$ conv

(b) $\displaystyle\int_1^\infty \frac{e^x}{\sqrt{1+x^2}}\,dx = \infty$ div.

$\to \infty$ $\dfrac{\cancel{\text{fast}}}{\text{slow}}$

6 (a) $\displaystyle\int_1^\infty \frac{2x^2-3x+\ln x}{2x-1}\,dx$

$\displaystyle \overset{LCT}{\approx} \int_1^\infty x\,.\,dx \approx \int_1^\infty \infty = \infty$ div.

(b) $\displaystyle\int_1^\infty \frac{e^{-x}}{x^3}\,dx = \int_1^\infty \frac{1}{x^3e^x}\,dx$

$x^3e^x > x^{100} \implies \dfrac{1}{x^3e^x} < \dfrac{1}{x^{100}} \implies$

$\int_1^\infty \frac{dx}{x^{100}}$ conv. (p-int, $p>1$) $\Rightarrow$ our int. $>$ conv.

$\Rightarrow$ conv. (DCT)

(e) $\int_1^\infty \frac{\ln x}{x^{1.5}}\,dx$

$$\ln x < x^{0.1} \Rightarrow \frac{\ln x}{x^{1.5}} < \frac{x^{0.1}}{x^{1.5}} = \frac{1}{x^{1.4}}$$

$\Rightarrow$ Our int. $< \int \frac{dx}{x^{1.4}}$ p-int. $p>1 \Rightarrow$ conv.

$\Rightarrow$ Our int $<$ conv. $\Rightarrow$ conv. (DCT)

(d) $$\int_{\boxed{0}}^{\pi} \frac{dx}{x+\sin x} = \lim_{t\to 0^+} \int_t^{\pi} \frac{dx}{x+\sin x}$$

on $[0,\pi]$ $0 < \sin x < 1$.

$\Rightarrow x < x+\sin x < x+1$

$\Rightarrow \frac{1}{x+1} < \frac{1}{x+\sin x} < \frac{1}{x}$

conv. $\qquad$ div.

In conclusive

Try: $0 < \sin x < x$ (tricky)

$$x < x + \sin x < 2x$$

$$\frac{1}{2x} < \frac{1}{x+\sin x} < \frac{1}{x}$$

Observe ~~that~~ $\int_0^{\pi} \frac{1}{2x}\,dx$ div.

$\Rightarrow$ our int $>$ div. $\Longrightarrow$ div.

$\overline{9}$ Sol : $\underset{3}{x^e}$, $\underset{4}{e^x}$, $\underset{5}{(\ln x)^x}$, $\overset{2}{\ln(x^x)}$, $\underset{1}{(\ln x)^2}$

$(\ln x)^2$: slowest.

$x^1 \ln x$: slower than x^e

$$(\ln x)^2 \quad (x \ln x) \quad x^e \quad e^x \quad (\ln x)^x$$

#10 $\int_1^{\infty} \frac{dx}{(x^2+1)^3}$

$$x^2+1 > x^2$$

$$(x^2+1)^3 > x^6$$

$$\frac{1}{(x^2+1)^3} < \frac{1}{x^6}$$

since $\int_1^{\infty} \frac{dx}{x^6}$ conv. $(p>1)$ $\Rightarrow$ Our int $<$ conv

$\Rightarrow$ conv. (DCT)